PART A

1) State three numbers between 0 and 1.

2) State four rational numbers between 3 and 4.

3) List all of the natural numbers from 1 through 5. Is this task easy to complete? Why or why not?

4) Try to list all of the rational numbers from 1 through 5? Is it possible to complete this task? Explain.

5) Can you list all of the real numbers from 1 through 5? Explain.

PART B

6) a) How many natural numbers are there from 1 through 10?
   b) How many whole numbers are there from 0 through 10?
   c) How many rational numbers are there from 1 through 10?
   d) How many irrational numbers are there from 1 through 10?
   e) How many real numbers are there from 1 through 10?

7) Explain what it means for a number set to have the density property.

8) Which of the following sets have the density property?
   - natural numbers
   - whole numbers
   - integers
   - rational numbers
   - irrational numbers
   - real numbers

9) Does the set of even integers have the density property? Explain.

10) The following sequences of numbers each have a *limit*. That is, they gradually get closer and closer to a specific number, called the *limit*. Identify the limit of each of the following sequences.

    a) $7.1, 7.01, 7.001, 7.0001, \ldots$ b) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ c) $\frac{5}{1}, \frac{5}{4}, \frac{5}{9}, \frac{5}{16}, \frac{5}{25}, \ldots$

    d) $0.3, 0.33, 0.333, 0.3333, \ldots$ e) $-2.6, -2.66, -2.666, -2.6666, \ldots$ f) $1.4, 1.44, 1.444, 1.4444, \ldots$

11) Consider the set $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$.

    a) Describe how the sequence of numbers in this set is constructed.
    b) Does this set have the density property? Explain.
    c) Does the sequence in this set have a limit? If so, what is the limit? If the sequence does not have a limit, explain why not.

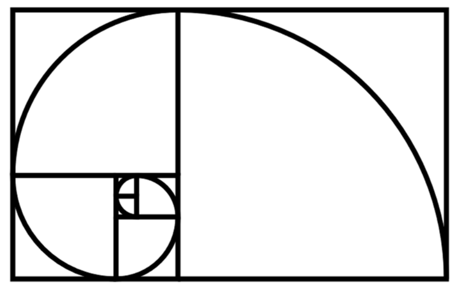
12) For each of the following expressions, imagine the value of n getting higher and higher. What happens to the value of the expression as n approaches infinity?

a) n b) $3n$ c) n^2 d) $\frac{1}{n}$ e) $\frac{1}{n^2}$ f) $\frac{n}{n+2}$ g) $\frac{n}{n^2-5}$ h) $-2n$

13) The *Fibonacci sequence* starts with 1 (or 0) and is followed by 1. From there, each term is found by adding the two previous terms, as shown below:

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

a) Does this sequence have a limit? Explain.
b) If you move through the sequence and divide each number by the previous number, the results approach a very famous number called the *golden ratio*. Determine this limit, rounded to five decimal places.
c) Instead of dividing each term by the previous term, if you divide each term by the next term, the results approach another value related to the golden ratio. Determine this value, rounded to five decimal places.
d) Do you notice anything interesting when you compare the values found in parts (b) and (c)? Explain.



*Can you see how the Fibonacci sequence and the golden ratio appear in this diagram of the **golden spiral**?*

PART C

14) Determine the limit of each sequence.

a) 0.2, 0.25, 0.252, 0.2525, ...

b) 3.128, 3.128128, 3.128128128, ...

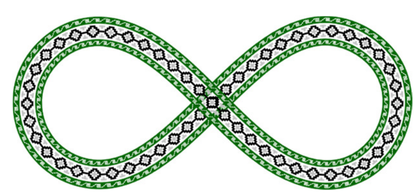
c) $\sqrt{2}, \sqrt{2\times\sqrt{2}}, \sqrt{2\times\sqrt{2\times\sqrt{2}}}, \sqrt{2\times\sqrt{2\times\sqrt{2\times\sqrt{2}}}}, \ldots$

15) Did you know it is possible to add infinitely many values and get a result that is not infinity? Consider the following sum.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

a) Hypothesize the value of this sum.
b) Use an area diagram to verify your hypothesis.

16) Even though there are infinitely many natural numbers, we say that they are *countable* (or *listable*) since we can list them in an order (1, 2, 3, 4, …). Hypothesize whether each of the following sets is countable or uncountable.
a) whole numbers
b) integers
c) rational numbers
d) real numbers



Infinite sets that are uncountable are said to represent a "higher level" of infinity than those that are countable.

ANSWERS

1) Answers will vary. For example, $\frac{1}{2}, \frac{2}{3}$, 0.2 .

2) Answers will vary. For example, $3\frac{1}{4}, \frac{29}{8}$, 3.5, 3.75 .

3) 1, 2, 3, 4, 5. The task is easy to complete because there are only five natural numbers from 1 through 5.

4) We cannot list all of the rational numbers from 1 to 5 because there are infinitely many.

5) We cannot list all of the real numbers from 1 to 5 because there are infinitely many.

6) a) 10 b) 11 c) infinitely many d) infinitely many e) infinitely many

7) A set has the density property if between any two numbers of the set there is another number that is also a member of the set. As a result, any two members of the set have infinitely many other members between them.

8) rational numbers, irrational numbers and real numbers

9) No. For example, there are no other even integers between 2 and 4.

10) a) 7 b) 0 c) 0 d) $\frac{1}{3}$ e) $-2\frac{2}{3}$ $\left(\text{or } -\frac{8}{3}\right)$ f) $1\frac{4}{9}$ $\left(\text{or } \frac{13}{9}\right)$

11) a) The sequence starts with 1 and then each successive term is found by dividing the previous term by 2.

b) No. For example, 1 and $\frac{1}{2}$ are both members of the set, but there is no other member of the set that falls between these two values since they are all less than $\frac{1}{2}$.

c) The sequence has a limit of 0.

12) a) approaches infinity b) approaches infinity c) approaches infinity d) approaches 0 e) approaches 0 f) approaches 1 g) approaches 0 h) approaches negative infinity

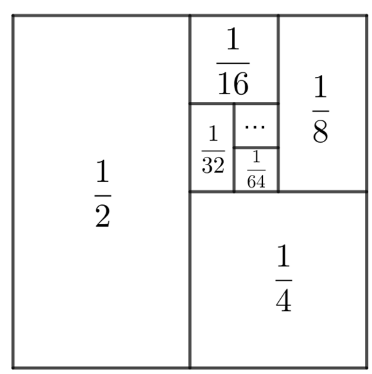
13) a) The sequence does not have a limit since its terms approach infinity.

b) 1.61803 c) 0.61803 d) The digits following the decimal point are the same.

14) a) $\frac{25}{99}$ b) $\frac{3125}{999}$ c) 2

15) a) 1

b) For the diagram on the right, the area of the entire square is 1.



16) a) countable

b) countable

c) countable

d) uncountable