TRANSFORMATIONS OF TRIGONOMETRIC FUNCTIONS

Transformations…again?!?

You have already seen most of the material that we are about to cover in this lesson! The major difference here, however, is that now we will focus on the use of radian measure when transforming the graphs of $y = \sin x$ and $y = \cos x$.

Recall the following transformations on the graph of $y = f(x)$:

Form	Transformation on the Graph of $y = f(x)$
$y = -f(x)$	• the graph of $y = f(x)$ is reflected in the *x*-axis (vertical reflection)
$y = f(-x)$	• the graph of $y = f(x)$ is reflected in the *y*-axis (horizontal reflection)
$y = af(x)$	• if $a > 1$, the graph of $y = f(x)$ is **vertically stretched** by a factor of a • if $0 < a < 1$, the graph of $y = f(x)$ is **vertically compressed** by a factor of a
$y = f(kx)$	• if $k > 1$, the graph of $y = f(x)$ is **horizontally compressed** by a factor of $\frac{1}{k}$ • if $0 < k < 1$, the graph of $y = f(x)$ is **horizontally stretched** by a factor of $\frac{1}{k}$

Let's consider a few examples in which the above transformations are applied to the functions $y = \sin x$ and $y = \cos x$.

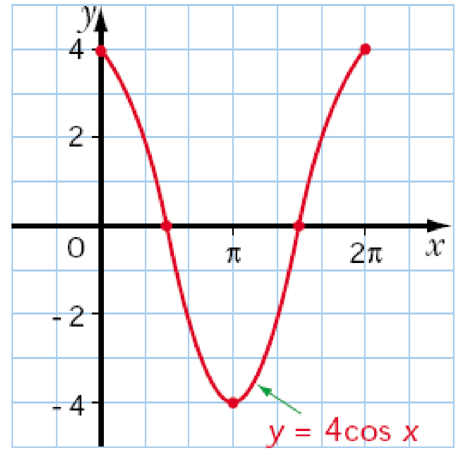
Before we move on, however, we should address the following question:

"How do I know if I should use radians or degrees when I'm solving these problems?"

- Look at what type of angle measurement is used in the question. If the question is given using degrees, then use degrees in your answer. If the question is given in radians, then answer in radians.

- **If the type of angle measurement is not given in the question, then <u>use radian measure</u>!**

Example 1
Sketch one cycle of the graphs of $y = 4\cos x$ and $y = \sin(3x)$, starting at $x = 0$. State the domain and range of the cycle in each case.

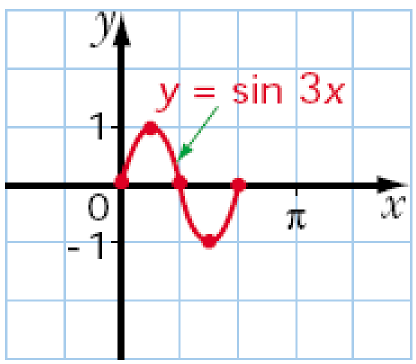


Solution

For $y = 4\cos x$:
The graph of $y = 4\cos x$ is simply the graph of $y = \cos x$ with a vertical stretch of factor 4. This stretch will increase the amplitude of 1 to 4, but will not affect the period of 2π.
As seen from the graph, the domain and range of the cycle are:

Domain: $\{x \in \mathbb{R} \mid 0 \le x \le 2\pi\}$

Range: $\{y \in \mathbb{R} \mid -4 \le y \le 4\}$



For $y = \sin(3x)$:
The graph of $y = \sin(3x)$ is the graph of $y = \sin x$ with a horizontal compression of factor $\frac{1}{3}$. This compression will decrease the period of 2π to $\frac{2\pi}{3}$, but will not affect the amplitude of 1. As seen from the graph, the domain and range of the cycle are:

Domain: $\left\{x \in \mathbb{R} \mid 0 \le x \le \frac{2\pi}{3}\right\}$

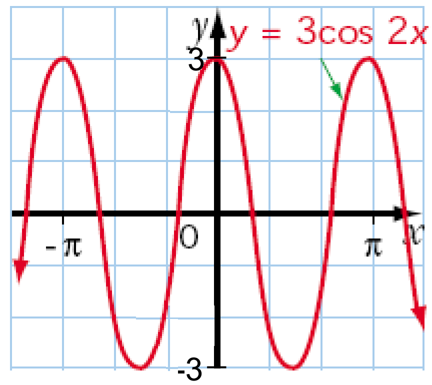
Range: $\{y \in \mathbb{R} \mid -1 \le y \le 1\}$

A couple of comments on the last example:

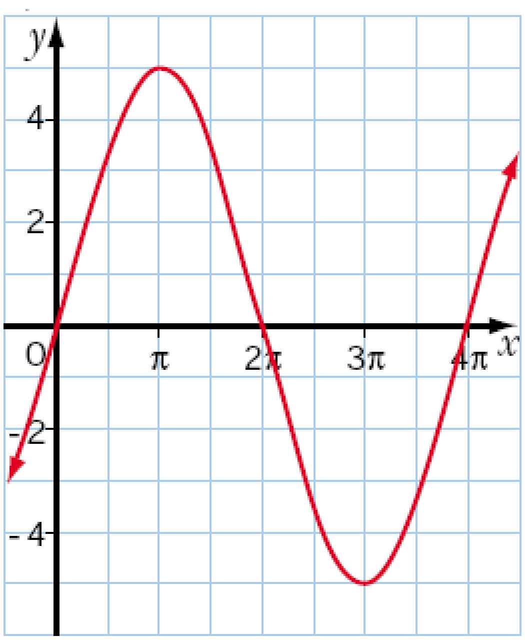
- Notice that vertical stretches/compressions affect the amplitude of the function, whereas horizontal stretches/compression affect the period.
- When transforming the graphs of $y = \sin x$ and $y = \cos x$, notice the key points that were used for reference (maxima, minima and x-intercepts)

Example 2
Sketch the graph of $y = 3\cos 2x$ over the domain $-\pi \le x \le \pi$.

Solution
The graph of $y = 3\cos 2x$ is the graph of $y = \cos x$ with a vertical stretch of factor 3 and a horizontal compression of factor $\frac{1}{2}$. Therefore, the original amplitude of 1 will increase to 3 and the original period of 2π will decrease to π.



Example 3
Determine the equation for the sine function with the given graph.



Solution
From the graph we see that the function has an amplitude of 5 and a period of 4π. Now, $y = \sin x$ has an amplitude of 1 and a period of 2π. Therefore, the given graph is the graph of $y = \sin x$ with a vertical stretch of factor 5 and a horizontal stretch of factor 2. The resulting equation is

$$y = 5\sin\left(\frac{1}{2}x\right)$$

A Couple More Transformations

Recall the following transformations on the graph of $y = f(x)$:

Form	Transformation on the Graph of $y = f(x)$
$y = f(x) + c$	• if $c > 0$, the graph of $y = f(x)$ is **translated (shifted)** c units upwards • if $c < 0$, the graph of $y = f(x)$ is **translated (shifted)** c units downwards
$y = f(x - d)$	• if $d > 0$, then $y = f(x)$ is **translated (shifted)** d units to the right • if $d < 0$, then $y = f(x)$ is **translated (shifted)** d units to the left

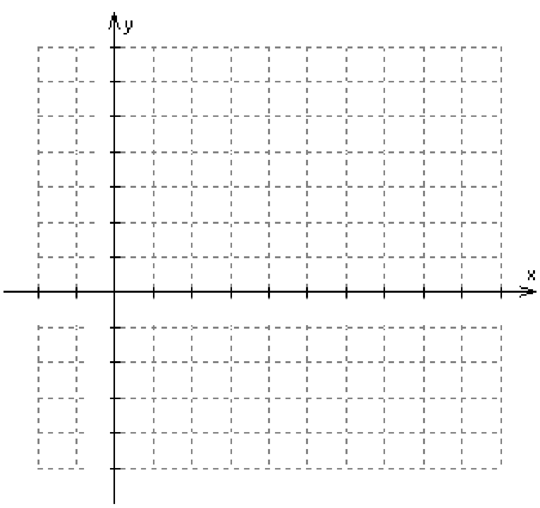
Let's consider a few examples in which the above transformations are applied to the functions $y = \sin x$ and $y = \cos x$. But first…

A Few Notes

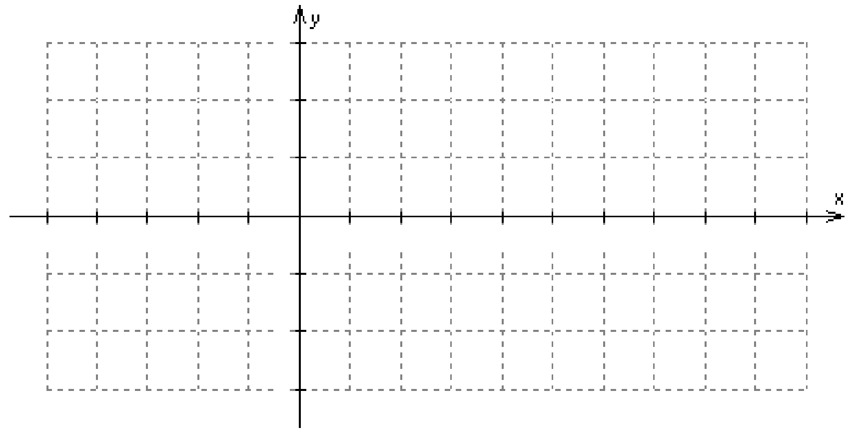
- For trigonometric functions, a horizontal translation is often called the **phase shift** or **phase angle**.

- When applying a combination of transformations to the graph of a function, remember that it is best to perform the reflections and stretches/compressions **before** performing the translations.

Example 4
Sketch one cycle of the graphs of $y = 2\sin x + 3$, starting at $x = 0$. State the domain and range of the cycle, as well as the equation of the axis.

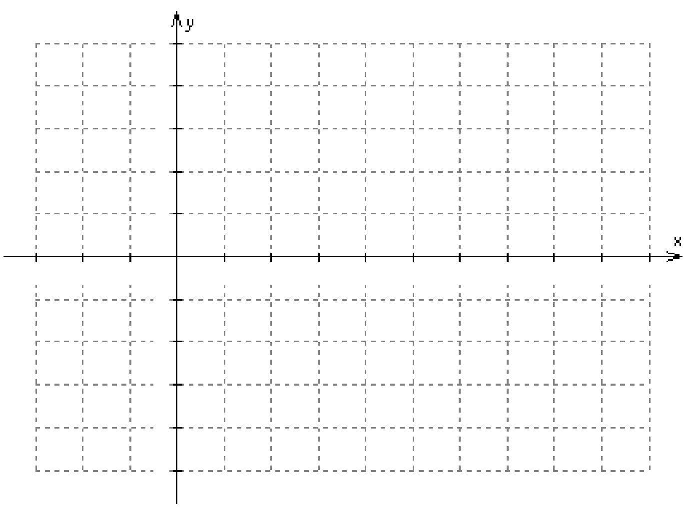


Example 5
State the amplitude, period, phase shift and equation of the axis for the function $y = 0.5\cos\left(x + \frac{\pi}{2}\right)$. Also, state the domain (for the entire function, not just one cycle) and range of the function. Sketch one cycle of the function's graph, starting at $x = 0$.



Example 6

Sketch the graph of $y = -4\sin 2\left(x - \frac{\pi}{2}\right)$ on the domain $0 \le x \le 2\pi$.



Example 7

Sketch the graph of $y = 3\cos\left(\frac{1}{2}x + \frac{\pi}{2}\right) - 1, \quad -4\pi \le x \le 4\pi$.

If necessary, remember to factor the coefficient of the *x*-term to identify the horizontal translations more easily.