DERIVATIONS OF TRIGONOMETRIC IDENTITIES

The Cosine Addition Formula

This formula is developed using a geometric diagram. A video walkthrough of its derivation can be found at **https://youtu.be/ipe7QJPoxmk**

The Cosine Subtraction Formula

BIG IDEA: Rewrite the subtraction as an addition and use the cosine addition formula.

$$\begin{aligned}\cos(x-y) &= \cos[x+(-y)] \\ &= \cos x\cos(-y) - \sin x\sin(-y) \\ &= \cos x\cos y - \sin x(-\sin y) \\ &= \cos x\cos y + \sin x\sin y\end{aligned}$$

The Sine Addition Formula

BIG IDEA: Rewrite $\sin(x+y)$ as $\cos\left[\frac{\pi}{2}-(x+y)\right]$, rearrange and use the cosine subtraction formula.

$$\begin{aligned}\sin(x+y) &= \cos\left[\frac{\pi}{2}-(x+y)\right] \\ &= \cos\left[\frac{\pi}{2}-x-y\right] \\ &= \cos\left[\left(\frac{\pi}{2}-x\right)-y\right] \\ &= \cos\left(\frac{\pi}{2}-x\right)\cos y + \sin\left(\frac{\pi}{2}-x\right)\sin y \\ &= \sin x\cos y + \cos x\sin y\end{aligned}$$

$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$

and

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$$

The Sine Subtraction Formula

BIG IDEA: Rewrite the subtraction as an addition and use the sine addition formula.

$$\begin{aligned}\sin(x-y) &= \sin[x+(-y)] \\ &= \sin x\cos(-y) + \cos x\sin(-y) \\ &= \sin x\cos y + \cos x(-\sin y) \\ &= \sin x\cos y - \cos x\sin y\end{aligned}$$

The Tangent Addition Formula

BIG IDEA: Use the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. After applying the sine and cosine addition formulas, divide the numerator and denominator by $\cos x \cos y$.

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)}$$

$$= \frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y}$$

Divide numerator and denominator by $\cos x \cos y$

$$= \frac{\dfrac{\sin x\cos y}{\cos x\cos y} + \dfrac{\cos x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y}{\cos x\cos y} - \dfrac{\sin x\sin y}{\cos x\cos y}}$$

$$= \frac{\dfrac{\sin x\cancel{\cos y}}{\cos x\cancel{\cos y}} + \dfrac{\cancel{\cos x}\sin y}{\cancel{\cos x}\cos y}}{\dfrac{\cancel{\cos x}\cancel{\cos y}}{\cancel{\cos x}\cancel{\cos y}} - \dfrac{\sin x\sin y}{\cos x\cos y}}$$

$$= \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

The Tangent Subtraction Formula

BIG IDEA: Rewrite the subtraction as an addition and use the tangent addition formula.

$$\tan(x-y) = \tan[x + (-y)]$$

$$= \frac{\tan x + \tan(-y)}{1 - \tan\tan(-y)}$$

$$= \frac{\tan x - \tan y}{1 - \tan x(-\tan y)}$$

$$= \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

The Sine Double Angle Formula

BIG IDEA: Rewrite $2x$ as $x+x$ and use the sine addition formula.

$$\begin{aligned}\sin 2x &= \sin(x+x)\\ &= \sin x\cos x+\cos x\sin x\\ &= 2\sin x\cos x\end{aligned}$$

The Cosine Double Angle Formulas

BIG IDEA: Rewrite $2x$ as $x+x$ and use the cosine addition formula. Use the Pythagorean identity $\sin^2 x+\cos^2 x=1$ to develop the other forms of the cosine double angle formula.

$$\begin{aligned}\cos 2x &= \cos(x+x)\\ &= \cos x\cos x-\sin x\sin x\\ &= \cos^2 x-\sin^2 x \quad ①\end{aligned}$$

$$\begin{aligned}&= \left(1-\sin^2 x\right)-\sin^2 x\\ &= 1-2\sin^2 x \quad ②\end{aligned}$$

$$\begin{aligned}&= \cos^2 x-\left(1-\cos^2 x\right)\\ &= \cos^2 x-1+\cos^2 x\\ &= 2\cos^2 x-1 \quad ③\end{aligned}$$

The Tangent Double Angle Formula

BIG IDEA: Rewrite $2x$ as $x+x$ and use the tangent addition formula.

$$\begin{aligned}\tan 2x &= \tan(x+x)\\ &= \frac{\tan x+\tan x}{1-\tan x\tan x}\\ &= \frac{2\tan x}{1-\tan^2 x}\end{aligned}$$

Alternate Forms of the Pythagorean Identity

BIG IDEA: Divide both sides of the Pythagorean identity $\sin^2 x+\cos^2 x=1$ by $\sin^2 x$ or $\cos^2 x$.

$$\begin{aligned}&\sin^2 x+\cos^2 x=1\\ &\frac{\sin^2 x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}=\frac{1}{\sin^2 x}\\ &1+\cot^2 x=\csc^2 x\\ &\therefore \csc^2 x=1+\cot^2 x\end{aligned}$$

$$\begin{aligned}&\sin^2 x+\cos^2 x=1\\ &\frac{\sin^2 x}{\cos^2 x}+\frac{\cos^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}\\ &\tan^2 x+1=\sec^2 x\\ &\therefore \sec^2 x=1+\tan^2 x\end{aligned}$$