NAME: _____

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KNOW / 12	APP / 12	INQ / 12	COMM / 6

MCV4U1 - UNIT 5 – DERIVATIVES OF EXPONENTIAL AND TRIGONOMETRIC FUNCTIONS

TEST

ROUND ALL ANSWERS TO THE NEAREST HUNDREDTH UNLESS OTHERWISE STATED

1) Differentiate each of the following and factor/simplify your answer as much as possible. (*K – 2, 2, 3, 3 marks*)

a) $f(x) = e^{\sin 3x}$

$f'(x) = e^{\sin 3x}\cos(3x)3$

$= 3e^{\sin 3x}\cos 3x$

b) $f(x) = \cos\left(e^x + x^2\right)$

$f'(x) = -\sin\left(e^x + x^2\right)\left(e^x + 2x\right)$

$= -\left(e^x + 2x\right)\sin\left(e^x + x^2\right)$

c) $f(x) = 3^{x^2}\cos(2x)$

$f'(x) = 3^{x^2}(\ln 3)(2x)\cos(2x) + (-\sin 2x)(2)3^{x^2}$

$= 2\left(3^{x^2}\right)\left[x(\ln 3)\cos 2x - \sin 2x\right]$

d) $f(x) = \dfrac{\tan\left(x^2\right)}{3^x}$

$f'(x) = \dfrac{\left(\sec^2 x^2\right)(2x)3^x - 3^x(\ln 3)\tan\left(x^2\right)}{\left(3^x\right)^2}$

$= \dfrac{3^x\left[2x\sec^2 x^2 - (\ln 3)\tan x^2\right]}{\left(3^x\right)^2}$

$= \dfrac{2x\sec^2 x^2 - (\ln 3)\tan x^2}{3^x}$

2) Determine, to the nearest hundredth, the slope of the tangent to $y = 5^{\cos x}$ where $x = \dfrac{\pi}{4}$. (*K – 2 marks*)

$y' = 5^{\cos x}(\ln 5)(-\sin x)$

$= -(\ln 5)(\sin x)5^{\cos x}$

For $x = \dfrac{\pi}{4}$,

$y' = -(\ln 5)\left(\sin\dfrac{\pi}{4}\right)5^{\cos\frac{\pi}{4}}$

$\doteq -3.55$

$\therefore$ the slope of the tangent is approximately -3.55.

3) Determine, to two decimal places, the coordinates of all local maximum and local minimum points for the function $f(x)=\dfrac{-2x^2-7x+13}{e^x}$. For each point, be sure to state whether it is a maximum or minimum that occurs. (*A - 4 marks*)

$$f(x)=\frac{-2x^2-7x+13}{e^x}$$

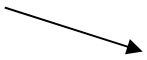
$$f'(x)=\frac{(-4x-7)e^x-e^x\left(-2x^2-7x+13\right)}{\left(e^x\right)^2}$$

$$=\frac{e^x\left(-4x-7+2x^2+7x-13\right)}{\left(e^x\right)^2}$$

$$=\frac{2x^2+3x-20}{e^x}$$

$$=\frac{(2x-5)(x+4)}{e^x}$$

Critical numbers are 2.5 and -4.

	$x<-4$	$-4<x<2.5$	$x>2.5$
Sign of $f'(x)$	+	-	+
Increase/Decrease for $f(x)$			

$\therefore$ There is a local maximum at (-4, 491.38)
and a local minimum at (2.5, -1.40)

4) Find the equation of the tangent line to $f(x)=x^2\sin x$ at the point where $x=\dfrac{\pi}{2}$. Leave all values *in exact form.* (*A - 4 marks*)

$$f(x)=x^2\sin x$$

$$f'(x)=2x\sin x+(\cos x)x^2$$

$$f'\left(\frac{\pi}{2}\right)=2\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2}+\left(\cos\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right)^2$$

$$=\pi$$

$$f\left(\frac{\pi}{2}\right)=\frac{\pi^2}{4}$$

For equation of tangent,

$$y=mx+b$$

$$y=\pi x+b$$

$$\frac{\pi^2}{4}=\pi\left(\frac{\pi}{2}\right)+b$$

$$\frac{\pi^2}{4}-\frac{\pi^2}{2}=b$$

$$b=-\frac{\pi^2}{4}$$

$\therefore$ the equation of the tangent is $y=\pi x-\dfrac{\pi^2}{4}$

5) Consider the function $f(x) = e^x \sin x$.

a) Find $f''(x)$. (*A – 2 marks*)

$f'(x) = e^x \sin x + e^x \cos x$

$$\begin{aligned} f''(x) &= e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x \\ &= 2e^x \cos x \end{aligned}$$

b) Determine the 20th derivative, $f^{20}(x)$, of *f*(*x*). (Hint: It is not necessary to actually differentiate 20 times!) (*A – 2 marks*)

$f'''(x) = 2\left(e^x \cos x - e^x \sin x\right)$

$$\begin{aligned} f^4(x) &= 2\left(e^x \cos x - e^x \sin x - e^x \sin x - e^x \cos x\right) \\ &= 2\left(-2e^x \sin x\right) \\ &= -4e^x \sin x \end{aligned}$$

Notice that $f^4(x) = -4f(x)$.

Therefore, $f^8(x) = (-4)^2 f(x)$

$= 16e^x \sin x$

Continuing this pattern, we get

$f^{20}(x) = (-4)^5 f(x)$

$= -1024e^x \sin x$

6) The number of rabbits in a forest is given by $N(t) = 30000\left(2^{-0.2t}\right)$, were *t* represents time in years and *N* represents the number of rabbits.

a) Determine, to the nearest tenth of a year, when the population will reach 10000. (*I – 3 marks*)

$$\begin{aligned} N(t) &= 30000\left(2^{-0.2t}\right) \\ \frac{1}{3} &= 2^{-0.2t} \\ \ln\left(\frac{1}{3}\right) &= \ln\left(2^{-0.2t}\right) \\ \ln\left(\frac{1}{3}\right) &= -0.2t \ln 2 \\ t &= \frac{\ln\left(\frac{1}{3}\right)}{-0.2 \ln 2} \\ t &\doteq 7.9 \end{aligned}$$

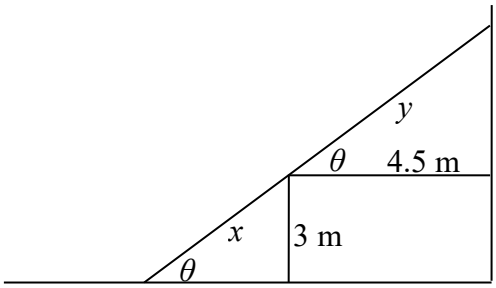
$\therefore$ the population will reach 10000 in approximately 7.9 years.

b) Find, to the nearest tenth, the rate of change of the rabbit population after 3 years. (*I – 3 marks*)

$$\begin{aligned} N(t) &= 30000\left(2^{-0.2t}\right) \\ N'(t) &= 30000\left(2^{-0.2t}\right)(\ln 2)(-0.2) \\ N'(3) &= 30000\left[2^{-0.2(3)}\right](\ln 2)(-0.2) \\ &\doteq -2743.8 \end{aligned}$$

$\therefore$ after three years, the rate of change is approximately -2743.8 rabbits/year.

7) A garage that is 3 m high is built against a very tall wall. The garage extends 4.5 m from the wall. Find, to the nearest hundredth of a meter, the length of the shortest ladder that can lean against the garage and touch the wall. (*I – 6 marks*)



$\sin\theta = \dfrac{3}{x}$ $\qquad$ $\cos\theta = \dfrac{4.5}{y}$

$x = \dfrac{3}{\sin\theta}$ $\qquad$ $y = \dfrac{4.5}{\cos\theta}$

$x = 3(\sin\theta)^{-1}$ $\qquad$ $x = 4.5(\cos\theta)^{-1}$

$l = x + y$

$= 3(\sin\theta)^{-1} + 4.5(\cos\theta)^{-1}$

$l' = -3(\sin\theta)^{-2}\cos\theta - 4.5(\cos\theta)^{-2}(-\sin\theta)$

$= \dfrac{-3\cos\theta}{\sin^2\theta} + \dfrac{4.5\sin\theta}{\cos^2\theta}$

For minimum,

$0 = \dfrac{-3\cos\theta}{\sin^2\theta} + \dfrac{4.5\sin\theta}{\cos^2\theta}$

$\dfrac{3\cos\theta}{\sin^2\theta} = \dfrac{4.5\sin\theta}{\cos^2\theta}$

$3\cos^3\theta = 4.5\sin^3\theta$

$\dfrac{3}{4.5} = \dfrac{\sin^3\theta}{\cos^3\theta}$

$\dfrac{3}{4.5} = \tan^3\theta$

$\theta \doteq 0.718 \ \ (41.140°)$

(angle must be in first quadrant)

Check for minimum:

	$0 < \theta < 0.718$	$0.718 < \theta < \dfrac{\pi}{2}$
Sign of l'	-	+
Increase/Decrease of l	↘	↗

$\therefore$ minimum

$\therefore l = 3(\sin 0.718)^{-1} + 4.5(\cos 0.718)^{-1}$

$= 10.5$

$\therefore$ The shortest ladder is approximately 10.5 m.

8) Find the derivative of $y = \csc x$. Express your answer using only reciprocal trigonometric ratios (secant, cosecant and cotangent). (*C – 3 marks*)

$y = \csc x$

$= (\sin x)^{-1}$

$y' = -(\sin x)^{-2}(\cos x)$

$= \dfrac{-\cos x}{\sin^2 x}$

$= -\left(\dfrac{\cos x}{\sin x}\right)\left(\dfrac{1}{\sin x}\right)$

$= -\cot x \csc x$

9) Bobby finds the derivative of a function and obtains the result $y' = e^x e^{-x^2}(x^2+1)(x-1)(\sin x + 5)$. He claims there is only one critical number, $x = 1$. Explain how Bobby is able to immediately arrive at such a conclusion. *(C – 3 marks)*

Bobby knows that the only factor that can achieve a value of 0 is the $(x – 1)$ factor. e^{anything} will only give positive values, $(x^2 + 1)$ will not give values less than 1 and $(\sin x + 5)$ will not give values less than 4. Therefore, none of the factors other than $(x – 1)$ will ever equal 0.