The Derivative of the General Exponential Function

When it comes to taking the derivative of exponential functions, we have only considered one very special exponential function…

$$f(x) = e^x$$

This function has the special property that it is its own derivative!

This property, however, is not true for exponential functions in general.

In this lesson, we will investigate the derivative of the general exponential function

$$f(x) = a^x, \text{ where } a > 0 \text{ and } a \neq 1$$

We will first investigate the derivatives of these functions by considering the function $f(x) = 2^x$ in the same way we dealt with the function $f(x) = e^x$.

Here we go!

$$f(x) = 2^x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h}$$

$$= \lim_{h \to 0} \frac{2^x\left(2^h - 1\right)}{h}$$

$$= 2^x \lim_{h \to 0} \frac{2^h - 1}{h}$$

$$= 2^x \lim_{h \to 0} \frac{2^h - 1}{h}$$

To find $\lim_{h \to 0} \frac{2^h - 1}{h}$, we can use tables.

h	$\frac{2^h - 1}{h}$
0.1	0.717734625
0.01	0.695555005
0.001	0.69338746
0.0001	0.6931712
0.00001	.693149

h	$\frac{2^h - 1}{h}$
-0.1	0.669670084
-0.01	0.690750455
-0.001	0.692907006
-0.0001	0.69312314
-0.00001	0.6931446

It appears that $\lim_{h \to 0} \frac{2^h - 1}{h} \doteq 0.693.$

So, we have

$$f'(x) = 2^x \lim_{h \to 0} \frac{2^h - 1}{h}$$

$$f'(x) = 2^x (0.693)$$

Hmmm…well, it appears that the graph of the derivative of $f(x) = 2^x$ is a vertical compression of the graph of $f(x) = 2^x$…

…but what is the significance of 0.693???

Think back to the way we found the derivative of $f(x) = e^x$.

$$f'(x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$f'(x) = e^x (1)$$

$$f'(x) = e^x$$

How does this 1 relate to our base of e?

Well, we know that $\log_e e = 1$.

In other notation, $\ln e = 1$.

So, back to $f(x) = 2^x$...

$$f'(x) = 2^x (0.693)$$

Maybe...just maybe, 0.693 is $\ln 2$...

IT IS!!!

$$f'(x) = 2^x (0.693)$$

So, $f'(x) = 2^x \ln 2$

In general,

If $f(x) = a^x$, where $a > 0$, then $f'(x) = a^x \ln a$.

<u>Formal Proof</u>

Using our knowledge of the derivative of $y = e^x$ and the fact that $a^x = e^{x \ln a}$, determine the derivative of the general exponential function $f(x) = a^x$ (on a separate page).

Some examples...

Differentiate each of the following.

a) $f(x) = 5^x$

$f'(x) = 5^x \ln 5$

b) $f(x) = 5^{2x-3}$

$f'(x) = 5^{2x-3} (\ln 5)(2)$

$= 2\left(5^{2x-3}\right) \ln 5$

Don't forget, we may have to use the product rule and the quotient rule with questions involving the derivative of exponential functions!