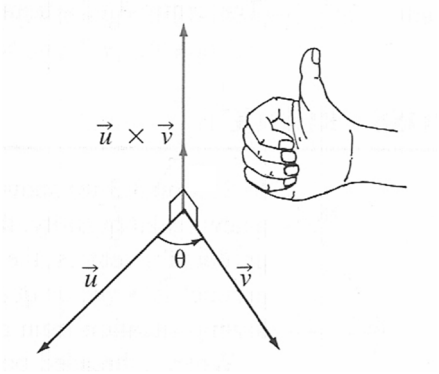
THE CROSS PRODUCT

Another Type of Vector Multiplication

Earlier, we became familiar with the dot product, a type of vector multiplication that results in a scalar. At this point, we will introduce another type of vector multiplication called the **cross product**, which has numerous applications in geometry and physics. Unlike the dot product, the cross product results in a vector quantity. We define the cross product as follows.

The Cross Product

The **cross product**, $\vec{u} \times \vec{v}$, of two vectors $\vec{u}$ and $\vec{v}$ is a vector with magnitude $\left|\vec{u}\right|\left|\vec{v}\right| \sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$ when arranged tail-to-tail. $\vec{u} \times \vec{v}$ acts perpendicular to both $\vec{u}$ and $\vec{v}$ in a direction given by the *right-hand rule*.

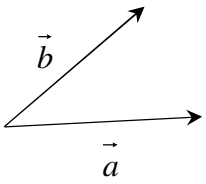


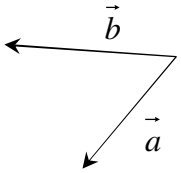
The Right-Hand Rule

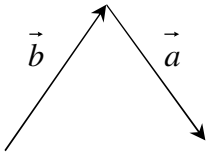
To determine the direction of $\vec{u} \times \vec{v}$, arrange the vectors tail-to-tail and curl the fingers of your right hand from $\vec{u}$ to $\vec{v}$. Your thumb points in the direction of $\vec{u} \times \vec{v}$.

Example

For each of the following pairs of vectors, state the direction of $\vec{a} \times \vec{b}$ (into the page or out of the page).

a) 

b) 

c) 

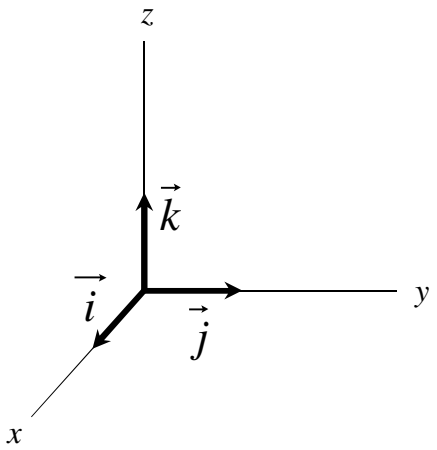
Can you think of any real world scenarios that relate to the definition of the cross product's direction?

Example

If $\left|\vec{u}\right| = 3$, $\left|\vec{v}\right| = 8$ and the angle between $\vec{u}$ and $\vec{v}$ is 30°, find $\left|\vec{u}\times\vec{v}\right|$ and $\left|\vec{v}\times\vec{u}\right|$. What can we say about the directions of $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$?

The Cross Product of Algebraic Vectors

To investigate the cross product of algebraic vectors, first consider the unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$, along with the definition of the cross product to determine the following cross products.



$\vec{i}\times\vec{i} =$	$\vec{i}\times\vec{j} =$	$\vec{i}\times\vec{k} =$
$\vec{j}\times\vec{j} =$	$\vec{j}\times\vec{i} =$	$\vec{j}\times\vec{k} =$
$\vec{k}\times\vec{k} =$	$\vec{k}\times\vec{i} =$	$\vec{k}\times\vec{j} =$

The Cross Product of Cartesian Vectors

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$. Then,

$$\vec{u}\times\vec{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

Example
If $\vec{u}$ = (1, 5, -3) and $\vec{v}$ = (-2, 1, 8), find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$. Is the cross product commutative?

An Easier Method

The expression for the cross product of algebraic vectors might be difficult to remember! The following method is much more convenient.

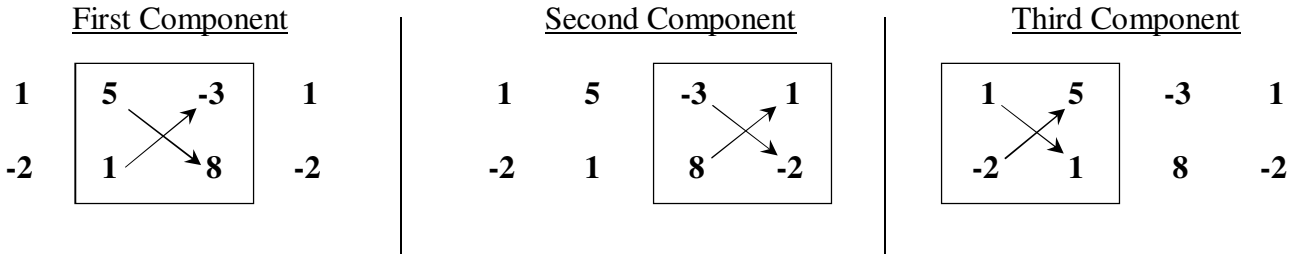
Example
If $\vec{u}$ = (1, 5, -3) and $\vec{v}$ = (-2, 1, 8), find $\vec{u} \times \vec{v}$.

- Write the components of the first vector ($\vec{u}$) in a row directly above the components of the second vector ($\vec{v}$), repeating the first component at the end of the row. **Since the cross product is not commutative, the order in which we write the vectors is important!!!**

1 5 -3 1

-2 1 8 -2

- Picture three boxes of numbers, as shown below. In each box, calculate the downward diagonal product minus the upward diagonal product. This process will give the components of the cross product $\vec{u} \times \vec{v}$. **Remember to start with the middle box!!!**



(5)(8) – (1)(-3) = 43 (-3)(-2) – (8)(1) = -2 (1)(1) – (-2)(5) = 11

Therefore, $\vec{u} \times \vec{v}$ = (43, -2, 11).

Example
Find $(2, -5, 4) \times (3, 1, 2)$.

Example
Find a vector that is perpendicular to (1, 2, 1) and (3, 0, 4). Verify that your solution is correct.

Are there any other possible solutions?

Notes About the Cross Product

- The cross product is a vector and is often called the **vector product**.
- The cross product is only defined in 3-space. Why?
- When calculating the cross product of two vectors, **ORDER MATTERS!!!**
- Note the use of "×" for the cross product and "•" for the dot product.

Properties of the Cross Product

The following properties can be proven using algebraic vectors and the definition of the cross product.

Properties of the Cross Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vectors in 3-space and let *k* be a scalar.

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ (Anti-commutative Law)
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ (Distributive Law)
- $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$
- $\vec{u} \times \vec{u} = \vec{0}$

Coplanar Vectors

Recall that the _____ gives a vector that is perpendicular to the two vectors being multiplied. Also, if the _____ is zero, then the two vectors being multiplied are perpendicular.

- How can we use these properties to determine if three given vectors are coplanar?

Test for Coplanar Vectors

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vectors in 3-space.

If _____, then $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar.

If _____, then $\vec{u}$, $\vec{v}$ and $\vec{w}$ are **not** coplanar.

Note: _____ is called the **triple scalar product** (or the scalar triple product) because it is a scalar quantity. Is it necessary to include the brackets when writing the scalar triple product?

Example

Determine if the vectors $\vec{a} = (1, 1, -2)$, $\vec{b} = (2, -3, 1)$ and $\vec{c} = (-1, -2, 3)$ are coplanar.

Example

Determine if the vectors $\vec{u} = (1, -1, -1)$, $\vec{v} = (2, 1, 4)$ and $\vec{w} = (-1, -1, -1)$ are coplanar.

Example

State whether the following expressions are vectors, scalars or meaningless.

a) $\vec{a} \bullet (\vec{b} \times \vec{c})$ _____

b) $(\vec{a} \bullet \vec{b}) \times (\vec{b} \bullet \vec{c})$ _____

c) $\vec{a} \times (\vec{b} \times \vec{c})$ _____

Example

Can we use the cross product to determine if two given vectors are collinear?