THE DOT PRODUCT

The Dot Product

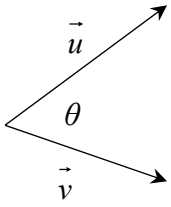
In our study of vectors, we have considered the operations of adding vectors, subtracting vectors and multiplying vectors by scalars. We will now consider the operation of multiplying two vectors.

There are actually two different types of vector multiplication that are useful in the application of vectors to problems in geometry and physics. The type of vector multiplication that we will consider first is the **dot product**.

The Dot Product

The dot product $\vec{u} \bullet \vec{v}$ is defined as follows:

$$\vec{u} \bullet \vec{v} = \left|\vec{u}\right|\left|\vec{v}\right| \cos\theta,$$



where θ is the angle between $\vec{u}$ and $\vec{v}$.

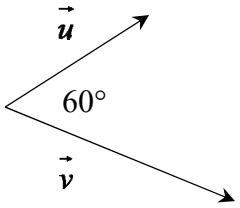
Important Notes about the Dot Product

- The dot product of two vectors is a scalar. For this reason, the dot product is also called the **scalar product**.

- We define the angle θ between two vectors to be the smaller (non-negative) angle formed when the vectors are arranged tail-to-tail (as shown in the diagram above). Therefore,
  - The dot product of perpendicular vectors is 0, since $\cos 90° = 0$.
  - Vectors forming an acute angle have a positive dot product, since the cosine of the angle is positive.
  - Vectors forming an obtuse angle have a negative dot product, since the cosine of the angle is negative.

Example

Calculate the dot product of $\vec{u}$ and $\vec{v}$ in the diagram below, where $\left|\vec{u}\right| = 4$ and $\left|\vec{v}\right| = 7$.



REMEMBER: The angle used for the dot product is that which is formed when the vectors are arranged tail-to-tail.

Properties of the Dot Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vectors and let k be a scalar.

1) For non-zero $\vec{u}$ and $\vec{v}$,
   a) if $\vec{u}$ and $\vec{v}$ are perpendicular, then $\vec{u} \bullet \vec{v} = 0$ (since $\cos 90° = 0$).
   b) if $\vec{u} \bullet \vec{v} = 0$, then $\vec{u}$ and $\vec{v}$ are perpendicular.

2) $\vec{u} \bullet \vec{u} = |\vec{u}|^2$

3) $\vec{u} \bullet \vec{0} = 0$

4) $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$ (Commutative Law)

5) $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$ (Distributive Law)

6) $(k\vec{u}) \bullet \vec{v} = k(\vec{u} \bullet \vec{v}) = \vec{u} \bullet (k\vec{v})$

Example

Expand and simplify $(2\vec{a} - \vec{b}) \bullet (\vec{a} + 3\vec{b})$.

Using the Dot Product to Find the Angle Between Two Vectors

By solving the dot product definition $\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|\cos\theta$ for $\cos\theta$, we obtain a useful formula for calculating the angle between two vectors.

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}||\vec{v}|}$$

NOTE: Instead of memorizing this formula, just remember how it is obtained from the definition of the dot product.

Example (complete on a separate page)

If the vectors $\vec{a} + 3\vec{b}$ and $4\vec{a} - \vec{b}$ are perpendicular, and $|\vec{a}| = 2|\vec{b}|$, determine the angle, to the nearest degree, between the nonzero vectors $\vec{a}$ and $\vec{b}$.